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Regularized Smoothing Approximations to Vertical Nonlinear Complementarity Problems

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1. INTRODUCTION

Given functions $F^1, \ldots, F^m$ from $\mathbb{R}^n$ into itself, the vertical nonlinear complementarity problem (VNCP), denoted by VNCP($F^1, \ldots, F^m$), is to find an $x \in \mathbb{R}^n$ such that

$$R(x) := \min\{x, F^1(x), \ldots, F^m(x)\} = 0, \tag{1}$$

where operator "min" is taken componentwise. When $m = 1$, VNCP reduces to the standard nonlinear complementarity problem (NCP), denoted by NCP(F), of finding an $x \in \mathbb{R}^n$ such that

$$r(x) := \min\{x, F(x)\} = 0.$$

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If we define $F\colon \mathbb{R}^n \to \mathbb{R}^n$ by

$$F(x) := \min\{F^1(x), \ldots, F^m(x)\}, \tag{2}$$

then VNCP$(F^1, \ldots, F^m)$ becomes NCP(F). Such a conversion not only establishes a clear relationship between NCP and VNCP but also motivates our investigation which enables us to obtain several interesting theoretical results (see Section 5). For NCP(F), there exist many efficient solution methods, but most of them depend on the differentiability of F, for example, the semismooth equation approach [8, 6, 17], and the smoothing approximation methods [3, 29, 4]. However, the min operator of F in (2) makes the function nondifferentiable in general. Unlike NCP, the study of VNCP has only begun (see [10, 12, 20]).

The motivation of the present study is two-fold. First, because the mountain pass theorem was introduced to the nonlinear complementarity problem by Facchinei and Kanzow [7], it has been discovered by Qi [23] that the theorem has algorithmic applications in proving the global convergence of a class of smoothing Newton methods for (box) variational inequality problems. It is our interest and purpose here to apply this theorem to the study of VNCP. Second, in two articles, aggregation function was used by Peng and Lin [21] to propose a noninterior point method for VLCP, and Qi and Liao [24] to introduce a smoothing Newton method for extended vertical LCP. Because previous results are only linear cases, we would like to address the nonlinear case in this paper. Upon the completion of this paper, we received a new report of Gowda and Tawhid [11]. The authors also use the aggregation function to reformulate VNCP as a P_0 equation. By applying results for P_0 functions to the reformulated system, they study the limiting behavior of the trajectory. They also obtain the similar result as our Theorem 3.4 (Section 3). However, our approach could easily lead to a numerical algorithm as we discuss in the final section.

In this paper, we first approximate $R(x)$ by a sequence (indexed by a parameter) of continuously differentiable functions, called aggregation functions (see Section 2). As a result, a sequence of approximation problems is created. Then a sequence of zero solutions of the approximated problems is obtained. The justification for such approximation is that the trajectory defined by the zero solutions of the approximated problems converge to a zero solution of $R(x)$ as the parameter approaches zero. The formulation of our aggregation functions is based on Tikhonov regularization, which is well used by several researchers in studying NCP and the box variational inequality problem, see [15, 7, 25, 23, 27]. For function $F(x)$ in (2), its Tikhonov regularization is defined as $F_t(x) :=$

$F(x) + tx$, where $t > 0$. Hence we can define Tikhonov regularization of $R(x)$ in (1) as

$$R(t, x) \coloneqq \min\{x, F^1(x) + tx, F^2(x) + tx, \ldots, F^m(x) + tx\}. \quad (3)$$

On top of $R(t, x)$, our aggregation function, which is defined in Section 2, can be formulated. The aggregation function which is parameterized in t leads to a sequence of approximation problems. As a result, the zero solutions of these approximation problems define a trajectory. In Section 3, we show that the sequence of above zero solutions converges to a solution of VNCP as $t \to 0$ under mild conditions. Furthermore, the existence, uniqueness, continuity, and the boundedness of the trajectory will be addressed. Section 4 is devoted to the discussion of ϵ-solution, which was originated in [26, 15], of VNCP$(F^1, \ldots, F^m)$. Finally in Section 5, a global error bound for VNCP with uniform P-property is obtained, the result extends the corresponding result of Chen and Harker [4] for NCP with uniform P-function.

In the remainder of the paper, we use the following notation:

Let $F(x)$ be defined in (2).

For $a \in \mathbb{R}^n$, let $[a]_+ \coloneqq \max\{0, a\}$, where the operator max is taken componentwise.

For a vector $u \in \mathbb{R}^n$, u_j denotes the jth component of u, $\nabla F^i(x)$ denotes the Jacobian matrix of $F^i(x)$, where $i \in \{1, \ldots, m\}$, and $j \in N \coloneqq \{1, \ldots, n\}$.

$F_j^i(x)$ denotes the jth component of $F^i(x)$, and $\nabla F_j^i(x)$ denotes the gradient of $F_j^i(x)$.

e is the vector of all ones in $\mathbb{R}^n$ and e_j denotes the jth column of the identity matrix.

2. MATHEMATICAL BACKGROUND

In this section, we state some mathematical results which are used in later discussions.

2.1. *Aggregation Function*

Let $g_1, \ldots, g_m: \mathbb{R}^n \to \mathbb{R}$ be given and

$$g(x) \coloneqq \max\{g_1, \ldots, g_m\}. \quad (4)$$

It is easy to see that g is piecewise continuously differentiable if each g_i is continuously differentiable. Let $t > 0$, the aggregation function $g(t, x)$ is defined by

$$g(t, x) := t \ln \sum_{i=1}^{m} \exp \frac{g_i(x)}{t}.$$

The function, viewed as an exponential penalty function, is studied intensively by Goldstein in [9]. When a multiplier method with an exponential penalty function was applied to $g(x)$, Bertsekas [1] obtained a slight variant of $g(t, x)$. Independently, the function was studied and named as the aggregation function by Li [19]. The exponential penalty has been employed by many other researchers as well, for example, Tseng and Bertsekas [30], and Kachiyan [18]. The following results for function $g(t, x)$ can be obtained.

LEMMA 2.1. *Assume that each g_i is continuously differentiable, then for any $t > 0$,*

(i)

$$0 \leq g(t, x) - g(x) \leq t \ln m. \tag{5}$$

(ii)

$$\nabla_x g(t, x) = \sum_{i=1}^{m} \lambda_i(t, x) \nabla g_i(x), \tag{6}$$

where

$$\lambda_i(t, x) = \frac{\exp(g_i(x)/t)}{\sum_{i=1}^{m} \exp(g_i(x)/t)} \in (0, 1),$$

$$i = 1, \ldots, m \quad \textit{and} \quad \sum_{i=1}^{m} \lambda_i(t, x) = 1. \tag{7}$$

The first result in Lemma 2.1 has been achieved in [2]. While the second one can be calculated directly.

Equation (5) indicates that $g(t, x)$ converges to $g(x)$ uniformly as t converges to zero. Notice that

$$R(t, x) = -\max\{-x, -F^1(x) - tx, \ldots, -F^m(x) - tx\},$$

by using Matlab notion, we denote

$$G(t,x) := -t\ln\left(\exp\frac{-x}{t} + \sum_{i=1}^{m}\exp\frac{-F^i(x)}{t-x}\right),$$

where the jth component of $G(t,x)$ is defined by

$$G_j(t,x) := -t\ln\left(\exp\frac{-x_j}{t} + \sum_{i=1}^{m}\exp\frac{-F^i_j(x)}{t-x_j}\right), \qquad j=1,\dots,n.$$

Then the similar result can be obtained for vector-valued function $G(t,x)$.

COROLLARY 2.2. *For any* $t > 0$, *we have*

$$R(t,x) - t\ln(m+1)e \le G(t,x) \le R(t,x). \tag{8}$$

Equation (8) is very useful. It reveals that the aggregation function $G(t,x)$ of $R(t,x)$ lies in the strip defined by $R(t,x)$ and $R(t,x) + t\ln(m+1)e$. It is obvious to see from (8) and (3) that $\lim_{t\to 0} G(t,x) = R(x)$. In Section 3, we study the existence of zero solution $x(t)$ of $G(t,x)$ and under what conditions $x(t)$ converges to a solution of $R(x) = 0$.

2.2. P_0- *and* P-*Functions*

DEFINITION 2.3. The mapping $f\colon \mathbb{R}^n \to \mathbb{R}^n$ is said to be a

- P_0-function if for every pair of distinct $x, y \in \mathbb{R}^n$, we have

$$\max_{x_i \ne y_i}(y-x)_i[f_i(y) - f_i(x)] \ge 0.$$

- P-function if for every pair of distinct $x, y \in \mathbb{R}^n$, we have

$$\max_{x_i \ne y_i}(y-x)_i[f_i(y) - f_i(x)] > 0.$$

- uniform P-function if for some $\gamma > 0$,

$$\max_{1\le i\le n}(y-x)_i[f_i(y) - f_i(x)] \ge \gamma\|x-y\|^2, \quad \textit{for all } x, y \in \mathbb{R}^n.$$

If f is a continuously differentiable P_0-function, then the Jacobian matrix $\nabla f(x)$ is a P_0-matrix at any point x. We recall that a matrix $M \in \mathbb{R}^{n\times n}$ is a P_0-matrix if and only if its every principal minor is nonnegative [5]. Let $F, F^1, \dots, F^m$ be functions from $\mathbb{R}^n$ to itself, we say F is a row representative of $\{F^1, \dots, F^m\}$ if for each $x \in \mathbb{R}^n$ and $i \in N$,

$F_i(x) \in \{F_i^1(x), \dots, F_i^m(x)\}$. We say $\{F^1, \dots, F^m\}$ has P_0-property (P-property, uniform P-property) if every representative of $\{F^1, \dots, F^m\}$ is a P_0-function (P-function, uniform P-function). Let $M_1, \dots, M_k \in \mathbb{R}^{n\times n}$. A matrix $M \in \mathbb{R}^{n\times n}$ is called a row representative of $(M_1, \dots, M_k)$ if for each $j = 1, 2, \dots, n$, the jth row of M is the jth row of some M_i, $i = 1, 2, \dots, k$. The following result for the row representative matrices has been obtained in Theorem 6 of [28].

LEMMA 2.4. *Let $M_1, \dots, M_k \in \mathbb{R}^{n\times n}$ be given, if determinants of all row representative matrices of $(M_1, \dots, M_k)$ are nonnegative and there is at least one such determinant which is positive. Then for arbitrary positive diagonal matrices $X_1, \dots, X_k \in \mathbb{R}^{n\times n}$,*

$$\det(X_1 M_1 + \cdots + X_k M_k) \neq 0.$$

A direct consequence of Lemma 2.4 is the following.

COROLLARY 2.5. *If $(F^1, \dots, F^m)$ has P_0-property and each F^i is continuously differentiable, then for arbitrary positive diagonal matrices $X_0, X_1, \dots, X_m \in \mathbb{R}^{n\times n}$,*

$$\det\left(X_0 + X_1 \nabla F^1(x) + \cdots + X_m \nabla F^m(x)\right) \neq 0.$$

Proof. It is easy to see that every row representative matrix of $(\nabla F^1(x), \dots, \nabla F^m(x))$ is the Jacobian matrix of some row representative of $(F^1(x), \dots, F^m(x))$. The observation implies that every row representative matrix is a P_0-matrix. Then the matrix block $(I, \nabla F^1(x), \dots, \nabla F^m(x))$ satisfies the condition of Lemma 2.4. The result follows directly. ∎

2.3. *Mountain Pass Theorem*

The following result is called the mountain pass theorem (see Theorem 5.3 in [7]), which is also well used in [23].

THEOREM 2.6. *Let $f: \mathbb{R}^n \to \mathbb{R}$ be continuously differentiable and coercive, i.e.,*

$$\lim_{\|x\| \to \infty} f(x) = +\infty.$$

Let $C \subset \mathbb{R}^n$ be a nonempty and compact set and define κ to be the least value of f on the (compact) boundary of C,

$$\kappa := \min_{x \in \partial C} f(x).$$

Assume further that there are two points $p \in C$ and $q \notin C$ such that $f(p) < \kappa$ and $f(q) < \kappa$. Then there exists a point $s \in \mathbb{R}^n$ such that $\nabla f(s) = 0$ and $f(s) \geq \kappa$.

3. BEHAVIOR OF THE TRAJECTORY

In this section, we establish the existence, uniqueness, boundedness, and continuity of the trajectory defined by the solutions of $G(t, x) = 0$. But first, we need to exploit some properties of $G(t, x)$. It has been proven by Ravindran and Gowda [25] that the composition of a $P_0(P)$-function and the min operator is also a $P_0(P)$-function. So if $\{F^1, \dots, F^m\}$ has P_0-property, the function $R(t, x)$ is a P-function for any $t > 0$ because $F(x) + tx$ is a P-function for $t > 0$ if F is a P_0-function (see Lemma 3.2 [7]). Hence $R(t, x)$ is univalence for $t > 0$. Next we prove $R(t, x)$ is coercive under the same condition.

PROPOSITION 3.1. *If $\{F^1, \dots, F^m\}$ has P_0-property, then $R(t, x)$ is coercive for every $t > 0$, i.e.,*

$$\lim_{\|x\| \to \infty} \| R(t, x) \| = \infty.$$

Proof. Assume the contrary that there exists an unbounded sequence $\{x^k\}$ such that $\{R(t, x^k)\}$ is a bounded sequence. Then there exists a subsequence $\{x^{k_l}\} \subset \{x^k\}$ and a row representative F of $\{F^1, \dots, F^m\}$ such that

$$R(t, x^{k_l}) = \min\{x^{k_l}, F(x^{k_l}) + tx^{k_l}\}.$$

Because $F(x)$ is a P_0-function, it follows from Proposition 3.4 [7] that there exists an index $i \in N$ and a subsequence $\{x^{k_j}\} \subset \{x^{k_l}\}$ such that

$$\left[x_i^{k_j} \to \infty, F_i(x^{k_j}) > -\infty\right] \quad \text{or} \quad \left[x_i^{k_j} \to -\infty, F_i(x^{k_j}) < \infty\right].$$

For both cases

$$\left| R_i(t, x^{k_j}) \right| \to \infty,$$

which contradicts the boundedness of $\{R(t, x^k)\}$. This establishes the result. ∎

Combining the results in Proposition 3.1 and (8), we have the following corollary.

COROLLARY 3.2. *If* $\{F^1, \dots, F^m\}$ *has* P_0*-property, then for each* $t > 0$, $G(t, x)$ *is coercive, that is,* $\lim_{\|x\| \to \infty} \|G(t, x)\| = \infty$.

Next, we explore the structure of Jacobian matrix $G(t, x)$ with respect to variable x. First let us look at the gradient of the jth component of $G(t, x)$,

$$\nabla_x G_j(t, x) = \lambda_0^j(t, x) e_j + \sum_{i=1}^{m} \lambda_i^j(t, x) \nabla F_j^i(x),$$

where

$$\lambda_0^j(t, x) = \frac{\exp(-x_j)/t}{\exp(-x_j)/t + e_j(t, x)} + \frac{e_j(t, x)}{\exp(-x_j)/t + e_j(t, x)} t,$$

$$\lambda_i^j(t, x) = \frac{\exp\left(-F_j^i(x)/t - x_j\right)}{\exp(-x_j)/t + e_j(t, x)},$$

$$e_j(t, x) = \sum_{l=1}^{m} \exp \frac{-F_j^l(x)}{t - x_j}.$$

Let

$$\Lambda_i(t, x) := \operatorname{diag}\left(\lambda_i^1(t, x), \dots, \lambda_i^n(t, x)\right), \qquad i = 0, 1, \dots, m.$$

Then it is easy to see that

$$\nabla_x G(t, x) = \Lambda_0(t, x) + \sum_{i=1}^{m} \Lambda_i(t, x) \nabla F^i(x).$$

Therefore, we have the following result for $G(t, x)$.

PROPOSITION 3.3. *Suppose that* $\{F^1, \dots, F^m\}$ *has* P_0*-property, then for any* $t > 0$, $G(t, x)$ *is a local homeomorphism.*

Proof. Noticing that $\lambda_i^j(t, x) > 0$ for any $t > 0$ and $i = 1, \dots, m$, $j \in N$, it follows from Corollary 2.5 that $\nabla_x G(t, x)$ is nonsingular and is continuous with respect to both t and x. By the implicit function theorem, $G(t, x)$ is a local homeomorphism. ∎

Now we are ready to introduce the main result of this paper.

THEOREM 3.4. *Suppose that* $\{F^1, \dots, F^m\}$ *has* P_0*-property, let* $\mathscr{S}$ *denote the solution set of* VNCP$(F^1, \dots, F^m)$. *Then*

(i) *For each* $t > 0$, *the equation* $G(t, x) = 0$ *has a unique solution, say* $x(t)$.

(ii) *The mapping* $t \to x(t)$ *is continuous and differentiable at any* $t > 0$.

(iii) *If* $\mathscr{S}$ *is nonempty and bounded, then the path* $\mathscr{P}_{\bar{t}}$,

$$\mathscr{P}_{\bar{t}} := \{x(t) | t \in (0, \bar{t}]\}$$

is bounded for any given $\bar{t} > 0$.

Proof. From a classical result mentioned by Ravindran and Gowda [25] that a coercive local homeomorphism of $\mathbb{R}^n$ is a global homeomorphism of $\mathbb{R}^n$, Propositions 3.2 and 3.3 imply the function $G(t, x)$ is a global homeomorphism of $\mathbb{R}^n$ for any $t > 0$. This establishes (i). Because the Jacobian matrix $\nabla_x G(t, x(t))$ is nonsingular and continuous as stated in the proof of Proposition 3.3, the implicit function theorem implies the continuity and differentiability of the mapping $t \to x(t)$ at any $t > 0$. This establishes (ii).

Now we prove (iii). It follows from (8) and (3) that

$$R(x) - t\|x\|e - t\ln(m+1)e \le G(t, x) \le R(x) + t\|x\|e.$$

The preceding inequalities indicate that $G(t, x)$ converges to $R(x)$ uniformly as $t \to 0$ on any compact set $C \subset \mathbb{R}^n$. Because $\mathscr{S}$ is nonempty and bounded, there exists a compact set $C \subset \mathbb{R}^n$ with $\mathscr{S} \subset \operatorname{int} C$. Let p be an arbitrary point in $\mathscr{S}$, then $R(p) = 0$. Define

$$\bar{\kappa} := \min_{x \in \partial C} \|R(x)\|^2 > 0.$$

Let $\psi_t(x) = \|G(t, x)\|^2$. Assume that the path $\mathscr{P}_{\bar{t}}$ is unbounded for some $\bar{t} > 0$, then from (i) and (ii) of this theorem there exists a sequence $\{x(t^k)\}$ such that $\psi_{t^k}(x(t^k)) = 0$ and $\|x(t^k)\| \to \infty$ as $t^k \to 0$. We note that $\psi_t(x)$ converges to $\|R(x)\|^2$ uniformly on compact set C as $t \to 0$. Therefore for sufficiently small t^k, we have that $\psi_{t^k}(x(t^k)) = 0$, $x(t^k) \notin C$ and

$$\min_{x \in \partial C} \psi_{t^k}(x) \ge \tfrac{3}{4}\bar{\kappa} \quad \text{and} \quad \psi_{t^k}(p) \le \tfrac{1}{4}\bar{\kappa}. \tag{9}$$

$\psi_t(x)$ is coercive from Corollary 3.2 for any $t > 0$. Applying mountain pass theorem 2.6 to $\psi_{t^k}(x)$ with $q = x(t^k)$, we obtain the existence of a vector $s \in \mathbb{R}^n$ such that

$$\nabla\psi_{t^k}(s) = 0 \quad \text{and} \quad \psi_{t^k}(s) \ge \tfrac{3}{4}\bar{\kappa} > 0. \tag{10}$$

Noticing that $\nabla_x \psi_{t^k}(s) = 2\nabla_x G(t^k, s)^T G(t^k, s)$, the nonsingularity of $\nabla_x G(t^k, s)$ and the first equality in (10) yield $G(t^k, s) = 0$, which contradicts the second inequality of (10). This establishes (iii). ∎

Remark. Similar results as in Theorem 3.4 for NCP have been obtained in [7, 25]. However, our trajectory $x(t)$ is not only continuous (achieved in [7, 25]) but also differentiable. Our stronger result stems from the aggregation function in our smoothing approximation.

By extending (iii) of Theorem 3.4, we have the following result.

COROLLARY 3.5. *If $\{F^1, \ldots, F^m\}$ has P_0-property and if $\mathscr{S}$ is nonempty and bounded, then $\{x(t)\}_{t>0}$ has at least one accumulation point which is a solution of VNCP.*

Proof. It is obvious that any accumulation point of $\{x(t)\}$ as $t \to 0$ is a solution of VNCP. From Theorem 3.4, $\{x(t)\}$ is bounded as $t \to 0$. This establishes the result. ∎

4. ϵ-SOLUTIONS

It would be very important and useful to characterize when an approximation solution is an ϵ-solution of the original problem. Such consideration has been studied in [15, 26].

DEFINITION 4.1. Given $\epsilon > 0$, we say that x^* is an ϵ-solution of VNCP$(F^1, \ldots, F^m)$ if (a) $x^* \geq 0$, (b) $\|F(x^*) - [F(x^*)]_+\| \leq \epsilon$, and (c) $|\langle x^*, F(x^*)\rangle| \leq \epsilon$.

When $m = 1$, the definition reduces to the ϵ-solution concept of Isac [15] for nonlinear complementarity problems.

THEOREM 4.2. *Let $\epsilon > 0$. Suppose that VNCP$(F^1, \ldots, F^m)$ has a nonempty and bounded solution set $\mathscr{S}$ and $\{F^1, \ldots, F^m\}$ has P_0-property. Then there exists $\epsilon_*(\epsilon) > 0$ such that for every $t \in (0, \epsilon_*(\epsilon))$, the corresponding solution $x(t)$ of $G(t, x) = 0$ is an ϵ-solution of VNCP$(F^1, \ldots, F^m)$.*

Proof. First, we note that, from Theorem 3.4, the solution path $\mathscr{P}_\epsilon$ is bounded. Let $\kappa_0 > 1$ satisfy

$$\|x(t)\| \leq \kappa_0, \quad \text{for all } t \in (0, \epsilon].$$

Second, the boundedness of $\mathscr{P}_\epsilon$ and the continuity of each F^i, $i = 1, \ldots, m$ imply sets $\{F^i(x(t))\}_{t \in (0, \epsilon]}$, $i = 1, \ldots, m$ are bounded. Let $\kappa_1 > 0$ such that

$$\|F^i(x(t))\| \leq \kappa_1, \quad \text{for all } t \in (0, \epsilon], i = 1, \ldots, m.$$

It follows from (8) that

$$\min\{x(t), F^1(x(t)) + tx(t), \ldots, F^m(x(t)) + tx(t)\} \geq G(t, x(t)) = 0,$$

which yields that

$$x(t) \geq 0 \quad \text{and} \quad F^i(x(t)) \geq -tx(t), \qquad i = 1, \ldots, m. \tag{11}$$

Hence (a) in Definition 4.1 is valid.

Now let

$$\epsilon_*(\epsilon) := \frac{\epsilon}{n \max\{\kappa_0, \kappa_1\} \ln(m+1) + \kappa_0^2}.$$

By noting $\epsilon_*(\epsilon) < \epsilon$, the second inequality in (11) implies

$$\|F(x(t)) - [F(x(t))]_+\| \leq t\|x(t)\| \leq \epsilon, \quad \text{for all } t \in (0, \epsilon_*].$$

This proves (b) in Definition 4.1.

Now we prove (c) in Definition 4.1. The first inequality in (8) gives

$$\min\{x(t), F^1(x(t)) + tx(t), \ldots, F^m(x(t)) + tx(t)\} \leq t \ln(m+1)e.$$

Because $x(t) \geq 0$, we have

$$\min\{x(t), F^1(x(t)), \ldots, F^m(x(t))\} \leq t \ln(m+1)e,$$

this together with the second inequality in (11) yield for all $j \in N$,

$$-t(x_j(t))^2 \leq x_j(t) F_j(x(t)) \leq t \max\{\kappa_0, \kappa_1\} \ln(m+1).$$

Hence,

$$|\langle x(t), F(x(t))\rangle| \leq (n \max\{\kappa_0, \kappa_1\} \ln(m+1) + \kappa_0^2)t \leq \epsilon,$$

for all $t \in (0, \epsilon_*]$. This completes the proof. ∎

5. ERROR BOUNDS FOR VERTICAL NONLINEAR COMPLEMENTARITY PROBLEMS WITH UNIFORM *P*-PROPERTY

This section is mainly motivated by the work of Chen and Harker [4] for nonlinear complementarity problems. The goal of this section is to establish the error bound result for VNCP($F^1, \ldots, F^m$) similar to Theorem 3.18 of [4] for NCP. But first, we need the following preliminary results.

It is known (Lemma 3.17, [4]) for any $a, b, c, d \in \mathbb{R}$,

$$|\min\{a, b\} - \min\{c, d\}| \leq \max\{|a - c|, |b - d|\}. \tag{12}$$

The result can be easily generalized to the finite many elements case as follows:

LEMMA 5.1. *Let* $a_1, b_1, \ldots, a_l, b_l \in \mathbb{R}$, *then*

$$\left|\min\{a_1, \ldots, a_l\} - \min\{b_1, \ldots, b_l\}\right| \leq \max\{|a_1 - b_1|, \ldots, |a_l - b_l|\}. \quad (13)$$

Proof. From (12), we have

$$\begin{aligned} &\left|\min\{a_l, \ldots, a_l\} - \min\{b_1, \ldots, b_l\}\right| \\ &\quad = \left|\min\{a_1, \min\{a_2, \ldots, a_l\}\} - \min\{b_1, \min\{b_2, \ldots, b_l\}\}\right| \\ &\quad \leq \max\left\{|a_1 - b_1|, \left|\min\{a_2, \ldots, a_l\} - \min\{b_2, \ldots, b_l\}\right|\right\} \\ &\quad \leq \cdots \\ &\quad \leq \max\{|a_1 - b_1|, \ldots, |a_l - b_l|\}. \end{aligned}$$

∎

The following result is a direct consequence.

COROLLARY 5.2. *If* F^i, $i = 1, \ldots, m$ *are Lipschitz continuous with the common Lipschitz constant* L, *then* $F(x)$ *is also Lipschitz continuous with the same constant.*

Proof. Let $x, y \in \mathbb{R}^n$, then from (13),

$$\begin{aligned} \|F(x) - F(y)\| &= \left\|\min\{F^1(x), \ldots, F^m(x)\} - \min\{F^1(y), \ldots, F^m(y)\}\right\| \\ &\leq \max\left\{\|F^1(x) - F^1(y)\|, \ldots, \|F^m(x) - F^m(y)\|\right\} \\ &\leq L\|x - y\|. \end{aligned}$$

∎

Based on the previous results, we are able to obtain the following properties for function $F(x)$ and VNCP($F^1, \ldots, F^m$).

PROPOSITION 5.3. *If* $\{F^1, \ldots, F^m\}$ *has uniform P-property, then* $F(x)$ *is a uniform P-function.*

Proof. Assume $x, y \in \mathbb{R}^n$ with $x \neq y$. Let $\tilde{F}$ and $\bar{F}$ be two representatives of $\{F^1, \ldots, F^m\}$ such that

$$F(x) = \tilde{F}(x), \qquad F(y) = \bar{F}(y),$$

then

$$\tilde{F}_i(x) \leq \bar{F}_i(x), \qquad \bar{F}_i(y) \leq \tilde{F}_i(y),$$

and $\bar{F}$ and $\tilde{F}$ are uniform P-functions.

For the index $i \in N$ with $x_i > y_i$, we have

$$\begin{aligned}
&(x_i - y_i)(F_i(x) - F_i(y)) \\
&\quad = (x_i - y_i)\big(\tilde{F}_i(x) - \bar{F}_i(y)\big) \\
&\quad = (x_i - y_i)\big(\tilde{F}_i(x) - \tilde{F}_i(y) + \tilde{F}_i(y) - \bar{F}_i(y)\big) \\
&\quad = (x_i - y_i)\big(\tilde{F}_i(x) - \tilde{F}_i(y)\big) + (x_i - y_i)\big(\tilde{F}_i(y) - \bar{F}_i(y)\big) \\
&\quad \geq (x_i - y_i)\big(\tilde{F}_i(x) - \tilde{F}_i(y)\big). \qquad (14)
\end{aligned}$$

For the index $i \in N$ with $x_i < y_i$, we have

$$\begin{aligned}
&(x_i - y_i)(F_i(x) - F_i(y)) \\
&\quad = (x_i - y_i)\big(\tilde{F}_i(x) - \bar{F}_i(y)\big) \\
&\quad = (x_i - y_i)\big(\tilde{F}_i(x) - \bar{F}_i(x)\big) + (x_i - y_i)\big(\bar{F}_i(x) - \bar{F}_i(y)\big) \\
&\quad \geq (x_i - y_i)\big(\bar{F}_i(x) - \bar{F}_i(y)\big). \qquad (15)
\end{aligned}$$

Combining (14) and (15) together yields

$$\begin{aligned}
&\max_i (x_i - y_i)(F_i(x) - F_i(y) \\
&\quad \geq \max_i \Big(\max\Big\{(x_i - y_i)\big(\tilde{F}_i(x) - \tilde{F}_i(y)\big), (x_i - y_i)\big(\bar{F}_i(x) - \bar{F}_i(y)\big)\Big\}\Big) \\
&\quad \geq \gamma \|x - y\|^2.
\end{aligned}$$

Here positive constant γ is the common uniform P constant of $\tilde{F}$ and $\bar{F}$. Because there are only finite many representatives of $\{F^1, \ldots, F^m\}$, such γ can be found due to the uniform P-property of $\{F^1, \ldots, F^m\}$. This completes the proof. ∎

COROLLARY 5.4. *Suppose that* $\{F^1, \ldots, F^m\}$ *has uniform P-property, then VNCP*$(F^1, \ldots, F^m)$ *has a unique solution.*

Proof. Because F is a uniform P-function by Proposition 5.3, it follows from Theorem 3.9 in [13] that NCP(F) has a unique solution. This establishes the result. ∎

By using the similar technique of Chen and Harker [4] for complementarity problem with uniform P-function (Lemma 3.16 and Theorem 3.18, [4]), we can obtain the following error bound results for VNCP$(F^1, \ldots, F^m)$.

THEOREM 5.5. *Suppose* $\{F^1, \dots, F^m\}$ *has uniform P-property and each* F^i, $i = 1, \dots, m$ *is Lipschitz continuous. Then we have*

(i)

$$\frac{1}{\max\{L, 1\}} \| R(x) - R(y) \| \leq \|x - y\| \leq \frac{L + 1}{\gamma} \| R(x) - R(y) \|, \quad \text{for all } x, y \in \mathbb{R}^n,$$

where L *is the common Lipschitz constant for* F^i, $i = 1, \dots, m$ (*see Proposition* 5.2), *and* γ *is the constant defined in the uniform P-function of* $F(x)$.

(ii)

$$\frac{1}{\max\{L, 1\}} \| R(x) \| \leq \|x - x^*\| \leq \frac{L + 1}{\gamma} \| R(x) \|, \quad \text{for all } x \in \mathbb{R}^n,$$

where x^* *is the unique solution of VNCP*$(F^1, \dots, F^m)$.

The proof of Theorem 5.5 should be straightforward following the similar argument of Lemma 3.16, Theorem 3.18, and Proposition 3.19 in [4].

6. CONCLUDING REMARKS

The existence of zero solution to the approximation equation $G(t, x) = 0$ plays an important role in a continuation method presented in [4]. Their method can be easily extended to VNCP. We should point out that Chen and Harker [4] viewed t as a parameter, and in each iteration their method either reduces t by a constant factor η, i.e., $t^{k+1} = \eta t^k$, $\eta \in (0, 1)$ or leaves it unchanged. Another way to solve $G(t, x) = 0$ is proposed by Hotta and Yoshise [14], Qi, Sun, and Zhou [22], and Tseng [29]. Their methods viewed t as a variable, that is, t and x share the same role in $G(t, x)$. They all considered the following mapping,

$$H(t, x) := \begin{pmatrix} t \\ G(t, x) \end{pmatrix},$$

and proposed Newton-like methods. In each iteration, both t and x are updated based on their Newton-like methods. Qi, Sun, and Zhou's method was further extended to a regularized version by Qi [23]. Of course there are other ways to formulate $H(t, x)$. For example, Jiang [16] used the exponential function $e^t - 1$ instead of the identity function t in his study of NCP. Furthermore, Newton and Gauss–Newton methods are proposed by Jiang.

It should be emphasized that all these methods can be extended to VNCP via $G(t, x)$. Therefore, the nonsingularity of the Jacobian matrix of $G(t, x)$ at x when $t > 0$ becomes critical and essential. Our theoretical study and structural exploitation on the Jacobian matrix (Section 3) provide a standard way for both the convergence analysis and the development of numerical algorithms in the study of VNCP.

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